The Penrose Polynomial and some Variants

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Outline

The Original Polynomial

- In his work published in 1971 [Pen71], Roger Penrose proposed a diagrammatic way to view tensor contractions and reformulated different tensor identities and equations.

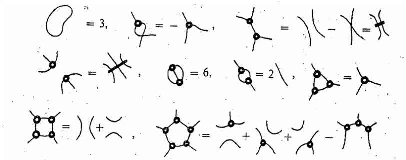


Figure: Various identities and relations from [Pen71]

The Original Polynomial

He observed that if $i\varepsilon_{abc}$ is put on the vertices, the resulting tensor contraction for planar graphs will be equal to the number of proper 3-edge colorings. The contraction can be calculated diagrammtically with the following relation:

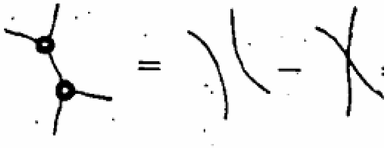


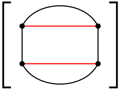
Figure: Penrose smoothings of an edge

The Original Polynomial

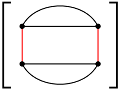
Among the many observations he made here are a few that are relevant to us:

- The "dimension" of a tensor can be extrapolated to negative values in the sense of formally setting contraction of the Kronecker delta with itself to $q \in \mathbb{Z}$.
- When evaluated at $\lambda = -2$ the value is proportional to the 3-colorings.
- This polynomial is related to the 4-color theorem (then a conjecture)
- The skein relation can only be applied to perfect matching edges (by a theorem of Petersen this exists for bridgeless graphs).
- For $\lambda = 3$ we can expand along a canonical matching replacing the vertices with triangles (blow up).

The Original Polynomial



$= 2\lambda(\lambda - 1)$



$= \lambda(\lambda - 1)^2$

The Original Polynomial

After Penrose's work, it was noted that the polynomial is to the transition polynomial of its medial graph. Later in 1997 Aigner [Aig97] defined and studied the construction and it's properties as a graph polynomial.

Definition (Penrose Polynomial)

The Penrose Polynomial $P(G, \lambda)$ for a graph G, is defined through the following relations on it's canonically checkerboard colored medial graph:

Figure: Skein Relation from [EM13b]

The Original Polynomial

Framed as a state sum, with the states being the "smoothed" vertices of the medial graph:

Definition (Penrose Polynomial State Sum)

The Penrose Polynomial $P(G, \lambda)$ for a graph G, is equal to the following sum over states s:

$$P(G, \lambda) = \sum_{s} (-1)^{cr(s)} \lambda^{c(s)}$$

where $cr(s)$ is the number of crossings and $c(s)$ is the number of loops in s.

The Original Polynomial

The following theorem can be found as proposition 4 of [Aig97] and references therein.

Theorem

The Penrose Polynomial counts the number of k-valuations, with $\lambda = k$.

The Original Polynomial

Definition (k-valuation)

Let i and j one of k colors on an edge with $i \neq j$. A k-valuation of a medial graph will have some i and j on each vertex satisfying:

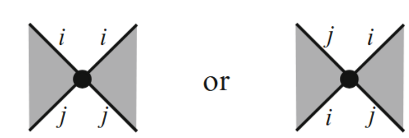
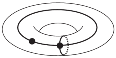
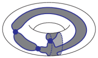


Figure: k-valuation from [EM13b]

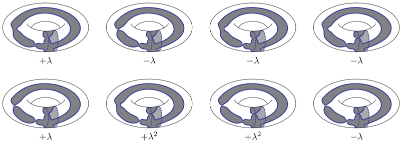
Heuristically, each k-valuation can be in a bijective manner to the loops of a colored state, such that each intersection or "kissing" has a different color.

An Approach for Embedded Graphs

In [EM13a] Ellis-Monaghan and Mofatt directly generalize the Penrose polynomial to embedded graphs on orientable surfaces, more or less following the same definition. For example, the state expansion of:

$G =$  $G_m =$ 

is



An Approach for Embedded Graphs

Although the generalization is straightforward, there are multiple important differences:

- There is a deletion-contraction formula, but it involves ribbon graph dualities(which can change the genus).
- We can express the polynomial as the sum over the chromatic polynomials of associated graphs.
- The polynomial does not generally count k-valuations.

An Approach for Embedded Graphs

Theorem

[EM13b] Theorem 4.28 The Penrose Polynomial of a ribbon graph can be expressed as the sum of chromatic polynomials over partial petrials:

$$P(G,\lambda) = \sum_{A \subseteq E(G)} (-1)^{|A|} \chi((G^{\tau(A)})^*, \lambda)$$

An Approach for Embedded Graphs

Theorem

[EM13b] Theorem 4.24 The Penrose Polynomial is can be expressed as the following sum over k-valuations:

$$P(G, \lambda) = \sum_{\text{k-valuation} \in s} \sum_{s} (-1)^{cr(s)}$$

Corollary

The Penrose Polynomial $P(G, \lambda)$ counts k-valuations for planar G when $\lambda = k$

From the bijection between k-valuations and loops, by the Jordan curve theorem $cr(s)$ is 1 for all k-valuations.

Another Approach for Embedded Graphs

For the non-planar case, we can find states where $cr(s)$ is odd:

odd # of virtual crossings in projection

Another Approach for Embedded Graphs

- Recently, Kauffman [Kau26] proposed a generalization called the "Penrose-Kauffman Polynomial" by viewing the virtual crossing as a tensor that "corrects "the negative sign in the sum.
- Formulating as a recursion relation results an independent virtual crossing and a singular crossing.*

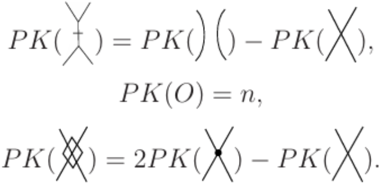


Figure: PK polynomial recursion relations [Kau26]

Another Approach for Embedded Graphs

The polynomial does not work on medial graphs and instead needs a perfect matching specified, counting the number of Tait k-colorings:

Definition (Penrose-Kauffman Polynomial)

The Penrose-Kauffman Polynomial $PK(\{G, M\}, \lambda)$ of graph with perfect matching is the polynomial that counts Tait k-colorings when $k = \lambda$

Another Approach for Embedded Graphs

Definition (Tait k-coloring)

A Tait k-coloring for a graph and perfect matching (G, M) are colorings of non-matching edges with k colors so that there exists colors $i \neq j$ satisfying the condition:

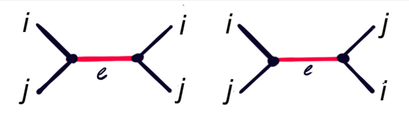


Figure: Tait k-coloring configuration from [KSW26]

Another Approach for Embedded Graphs

This polynomial can also be expressed as the sum of chromatic polynomials over associated graphs. The associated graphs are called component graphs G_S that are the graphs with vertices at link components in loop expansion with edges at smoothings.

Theorem

[KSW26] Theorem 3.4 The PK polynomial is the sum of chromatic polynomials over states:

$$PK(\{G, M\}, \lambda) = \sum_{S} \chi(G_S, \lambda)$$

It suffices to show that Tait k-colorings correspond to the edge coloring (chromatic polynomial) of some state graph.

Other Work

There is also other related to generalizations of the Penrose polynomial worth mentioning:

- Early work generalized the Penrose polynomial to binary matroids. See [Chu+14] and references therein which also talks about a version for delta matroids.
- Baldridge and his student McCarty constructed some cochain complex on the space of Penrose smoothings to find the Penrose-Kauffman polynomial as the Poincaré polynomial and a different polynomial called the "Baldridge-McCarty polynomial" that computes the Euler characteristic.

Penrose-Kauffman and Aigner's Polynomial

Notice that definitions of k-valuations and Tait k-colorings are very similar. This can be used to relate the polynomials:

Theorem ([KSW26])

The PK polynomial of the blow-up graph is the same as the Penrose polynomial of the original graph.

Proof sketch: Tait k-colorings on the blow-up graph with canonical perfect matchings are the same as k-valuations for the medial graph.

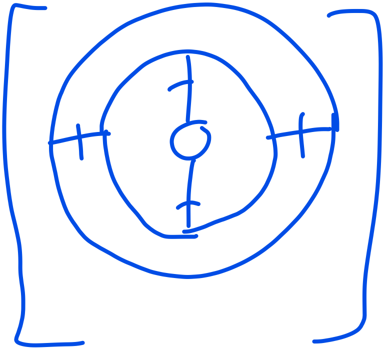
Penrose-Kauffman and Aigner's Polynomial

The Penrose-Kauffman Polynomial is more general than Aigner's Penrose Polynomial in the sense that there exists a polynomial $p(q)$ not equal

Lemma

[KSW26] Proposition 8.8 The linear coefficient of the Penrose polynomial in the plane is even.

Example 6.5 of [KSW26] gives a PK polynomial that has odd linear coefficient.



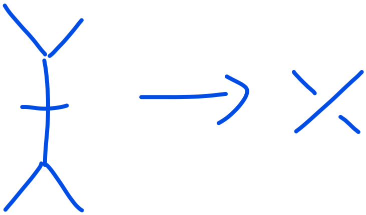
$$= -3q^3 - 2q^2 + 5$$

Penrose-Kauffman and Ellis-Monaghan-Mofatt's Generalization

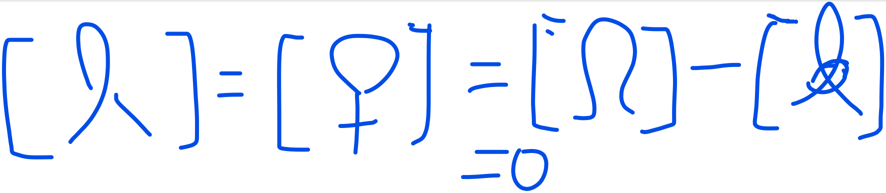
The same logic does not hold for to establish a relationship in the non-planar case since as we've seen before the Penrose polynomial does not count k-valuations of the ribbon graph.

Penrose Polynomials for Knots

A trivalent graph along with a perfect matching (G, M) can be associated with a knot with the following rule:



Penrose Polynomials for Knots



This is not invariant under the Reidemeister moves! It can however be well-defined as an invariant for alternating knots.

Penrose Polynomials for Knots

Tait famously conjectured that the reduced alternating diagrams of alternating knots can be taken to each other through a finite series of moves called *flypes*. This was later proved by Murasugi and Thistlethwaite and can be used to show that the Penrose-Kauffman polynomial associated with the diagram is well defined.

Theorem

[KSW26] Proposition 7.1 The Penrose-Kauffman polynomial of the perfect matching graph of a reduced alternating diagram is invariant under flypes.

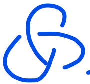
Penrose Polynomials for Knots

Definition (Coloring Polynomial)

The coloring polynomial of a alternating knot is the Penrose-Kauffman polynomial of the perfect matching graph of it's reduced diagram.

Corollary

The trefoil knot is chiral.

[] $= \lambda(\lambda-1)(\lambda-2)$, [] $= 4\lambda(\lambda-1)$

Penrose Polynomials for Knots

Here are a few questions related to the coloring polynomial:

- Can one show that it is always non-zero (proves 4-color theorem [KSW26])
- Is it an invariant for virtual or multi-virtual knots? (As far as I can tell, there is only a conjecture to virtual flypes with no counterexamples).
- How powerful is this invariant?

Duality Relations

In the planar case, we have a duality relation for the Penrose polynomial and PK Polynomial (for example Theorem 6.3 of [KSW26]. [EM13b] also includes many relations with generalized dualities for ribbon graphs. Do these, or something similar hold for the Penrose-Kauffman polynomial?

Differences between Polynomials

As we've seen before, the Penrose-Kauffman and Penrose polynomial do not have a straightforward relationship.

- For ribbon graphs, do there exist polynomials such that there is a Penrose-Kauffman polynomial but not Penrose or vice-versa?
- Can we quantify the difference between the Penrose-Kauffman polynomial of the blow-up of the ribbon graph and the Penrose polynomial?

Efficient Computation?

- Can we efficiently compute some points of the PK polynomial? (The number of perfect matchings for planar and some non-planar graphs can be efficiently computed through the Fischer-Kasteleyn-Tempereley algorithm).
- Like the Tutte polynomial this answer may differ at some points.

Coefficients and Zeroes

In [KSW] and [Aig97]the following conjectures were made pertaining to the coefficients of the Penrose/PK polynomial:

- The linear coefficient in the planar case is always non-zero. (Not true for surface).
- The coefficients of the Penrose polynomial is weakly alternating (alternating between ≤ 0 and ≥ 0). Implies 4-color theorem through the relation between $P(3)$ and $P(-2)$.

Coefficients and Zeroes

The zeroes of graph polynomials can also contain interesting information. Lee and Yang initiated this study by showing that the zeroes of the Ising model partition function (specialization of the Tutte polynomial) lie on the unit circle of $\mathbb{C}$

- What kind of pattern does the locus of zeroes make large scale limits?
- Is there information about the complexity in the zeroes?
- Is there information about the genus of the surface in the zeroes?

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